

A new method for constructing exact solutions to nonlinear delay partial differential equations

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Abstract

We propose a new method for constructing exact solutions to nonlinear delay reaction–diffusion equations of the form

$$u_t = ku_{xx} + F(u, w),$$

where $u = u(x, t)$, $w = u(x, t - \tau)$, and τ is the delay time. The method is based on searching for solutions in the form $u = \sum_{n=1}^N \xi_n(x) \eta_n(t)$, where the functions $\xi_n(x)$ and $\eta_n(t)$ are determined from additional functional constraints (which are difference or functional equations) and the original delay partial differential equation. All of the equations considered contain one or two arbitrary functions of a single argument. We describe a considerable number of new exact generalized separable solutions and a few more complex solutions representing a nonlinear superposition of generalized separable and traveling wave solutions. All solutions involve free parameters (in some cases, infinitely many parameters) and so can be suitable for solving certain problems and testing approximate analytical and numerical methods for nonlinear delay PDEs. The results are extended to a wide class of nonlinear partial differential-difference equations involving arbitrary linear differential operators of any order with respect to the independent variables x and t (in particular, this class includes the nonlinear delay Klein–Gordon equation) as well as to some partial functional differential equations with time-varying delay.

Keywords: delay partial differential equations; delay reaction-diffusion equations; exact solutions; generalized separable solutions; functional differential equations; time-varying delay; nonlinear equations; delay Klein–Gordon equations

1 Introduction

Nonlinear delay partial differential equations and systems of coupled equations arise in biology, biophysics, biochemistry, chemistry, medicine, control, climate model theory, ecology, economics, and many other areas (e.g., see the studies [1–11] and references in them). It is noteworthy that similar equations occur in the mathematical theory of artificial neural networks, whose results are used for signal and image processing as well as in image recognition problems [12–21].

The present paper deals with nonlinear delay reaction–diffusion equations [1, 3, 11, 22] of the form

$$u_t = ku_{xx} + F(u, w), \quad w = u(x, t - \tau). \quad (1)$$

A number of exact solutions to the heat equation with a nonlinear source, which is a special case of equation (1) without delay and with $F(u, w) = f(u)$, are listed, for example, in [23–29]. Most

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comprehensive surveys of exact solutions to this nonlinear equation can be found in the handbook [30]; it also describes a considerable number of generalized and functional separable solutions to nonlinear reaction–diffusion systems of two coupled equations without delay.

The list of known exact solutions to equation (1) is quite limited.

In general, equation (1) admits traveling-wave solutions, $u = u(\alpha x + \beta t)$. Such solutions are dealt with in many studies (e.g., see the studies [2–7] and references in them).

A complete group analysis of the non-linear differential-difference equation (1) was carried out in [11]. Four equations of the form (1) were found to admit invariant solutions; two of these equations are of limited interest, since they have degenerate solutions (linear in x). There was only one equation that involved an arbitrary function and had a non-degenerate solution:

$$u_t = ku_{xx} + u[-a \ln u + f(wu^{-b})], \quad b = e^{a\tau}, \quad (2)$$

where $f(z)$ is an arbitrary function. The exact solution to this equation found in [11] was

$$u = \exp(Cxe^{-at})\varphi(t), \quad (3)$$

where C is an arbitrary constant and $\varphi(t)$ is a function satisfying the delay ordinary differential equation

$$\varphi'(t) = \varphi(t)[C^2ke^{-2at} - a \ln \varphi(t) + f(\varphi(t-\tau)\varphi^{-b}(t))]. \quad (4)$$

The other equation obtained in [11] that had a non-degenerate solution coincides, up to notation, with a special case of equations (4), at $f(z) = c_1 + c_2 \ln z$.

Remark 1. It is noteworthy that equation (2) is explicitly dependent on the delay time τ , which corresponds to a more general kinetic function $F(u, w, \tau)$ than in equation (1). It is only at $a = 0$ in (2) that we have a kinetic function explicitly independent of τ : $F(u, w) = uf(w/u)$; in this case, (3) represents a separable solution, $u = e^{Cx}\varphi(t)$.

In what follows, the term ‘exact solution’ with regard to nonlinear partial differential-difference equations, including delay partial differential equations, is used in the following cases:

- (i) the solution is expressible in terms of elementary functions or in closed form with definite or indefinite integrals;
- (ii) the solution is expressible in terms of solutions to ordinary differential or ordinary differential-difference equations (or systems of such equations);
- (iii) the solution is expressible in terms of solutions to linear partial differential equations.

Combinations of cases (i)–(iii) are also allowed.

This definition generalizes the notion of an exact solution used in [30] with regard to nonlinear partial differential equations.

Remark 2. Solution methods and various applications of linear and nonlinear ordinary differential-difference equations, which are much simpler than nonlinear partial differential-difference equations, can be found, for example, in [31–36].

2 General description of the functional constraints method

Consider a wide class of nonlinear delay reaction–diffusion equations:

$$\begin{aligned} u_t &= ku_{xx} + uf(z) + wg(z) + h(z), \\ w &= u(x, t - \tau), \quad z = z(u, w), \end{aligned} \quad (5)$$

where $f(z)$, $g(z)$, and $h(z)$ are arbitrary functions and $z = z(u, w)$ is a given function. In addition, we will sometimes consider more complex equations where f , g , and h can additionally depend on the independent variables x or/and t explicitly.

We look for generalized separable solutions of the form

$$u = \sum_{n=1}^N \Phi_n(x) \Psi_n(t), \quad (6)$$

where the functions $\Phi_n(x)$ and $\Psi_n(t)$ are to be determined in the analysis.

Remark 3. For nonlinear partial differential equations, various modifications of the method of generalized separation of variables based on searching for solutions of the form (6) are detailed, for example, in [27, 28, 30]. These studies also present a large number of equations that admits generalized separable solutions.

For nonlinear delay partial differential equations of the form (5) that involve arbitrary functions, the direct application of the method of generalized separation of variables turns out to be ineffective.

The new approach pursued in the present paper is based on searching for generalized separable solutions of the form (6) that satisfy one the following two additional *functional constraints*:

$$z(u, w) = p(x), \quad w = u(x, t - \tau); \quad (7)$$

$$z(u, w) = q(t), \quad w = u(x, t - \tau). \quad (8)$$

These constraints represent difference equations in t with x playing the role of a free parameter. The function $z = z(u, w)$ appears in equation (5) as the argument of the arbitrary functions. The functions $p(x)$ and $q(t)$ are implicitly dependent on x and t (expressible in terms of $\Phi_n(x)$ and $\Psi_n(t)$, respectively) and are determined from the analysis of equations (7) or (8) taking into account (6). It should be emphasized that there is no need to obtain general solutions of equations (7) or (8); particular solutions will suffice.

A solution to the difference equation (7) or (8), in view of (6), determines an allowed form of the exact solution, whose final representation is subsequently obtained by substituting the resulting solution into the specific equation (5).

In what follows, constraints (7) and (8) will be referred to as a *functional constraint of the first kind* and *functional constraint of the second kind*, respectively.

For functional constraint of the first kind (7), the functions $\Psi_n(t)$ appearing in (6) are usually chosen in the form

$$\begin{aligned} \Psi_0(t) &= 1, \quad \Psi_1(t) = t, \quad \Psi_n(t) = e^{\lambda_n t}, \\ \Psi_n(t) &= e^{\lambda_n t} \cos(\beta_n t), \quad \Psi_n(t) = e^{\lambda_n t} \sin(\beta_n t), \end{aligned} \quad (9)$$

with the parameters λ_n and β_n determined from (7).

Remark 4. We have introduced the term *functional constraint* by analogy with the term *differential constraint*, which is employed in the method of differential constraints used to seek exact solutions to nonlinear partial differential equations; for a description of this method and application examples, see, for example, [30, 37–39]). In more complex cases, the original equation and functional constraints (7) and (8) can contain a time-varying delay $\tau = \tau(t)$ instead of the constant delay τ ; see the comments after equation (83) in Section 8.

Below we give examples of applying the above method to constructing generalized separable solutions to some equations of the form (5) as well as more complex nonlinear partial differential equations with delay.

3 The equation contains one arbitrary function dependent on w/u

Equation 1. Consider the equation

$$u_t = k u_{xx} + u f(w/u), \quad (10)$$

which is a special case of equation (5) with $g = h = 0$ and $z = w/u$.

1.1. The functional constraint of the second kind (8) becomes

$$w/u = q(t), \quad w = u(x, t - \tau). \quad (11)$$

It is clear that the difference equation (11) can be satisfied with a simple separable solution

$$u = \varphi(x)\psi(t), \quad (12)$$

which gives $q(t) = \psi(t - \tau)/\psi(t)$. Substituting (12) into (10) and separating the variable, we arrive at the following equations for $\varphi(x)$ and $\psi(t)$:

$$\varphi'' = a\varphi, \quad (13)$$

$$\psi'(t) = ak\psi(t) + \psi(t)f(\psi(t - \tau)/\psi(t)), \quad (14)$$

where a is an arbitrary constant.

The general solution of equation (13) is expressed as

$$\varphi(x) = \begin{cases} C_1 \cos(\sqrt{|a|}x) + C_2 \sin(\sqrt{|a|}x) & \text{if } a < 0; \\ C_1 \exp(-\sqrt{a}x) + C_2 \exp(\sqrt{a}x) & \text{if } a > 0; \\ C_1 x + C_2 & \text{if } a = 0, \end{cases} \quad (15)$$

where C_1 and C_2 are arbitrary constants.

Remark 5. Simple separable solutions of the form (12) are also admitted by the more general equation

$$u_t = ku_{xx} + uf(t, w/u),$$

in which the kinetic function is additionally dependent on t .

1.2. In the simple case $p(x) = p_0 = \text{const}$, the functional constraint of the first kind (7) for equation (10) is written as

$$w/u = p_0, \quad w = u(x, t - \tau). \quad (16)$$

Solutions of the difference equation (16) with $p_0 > 0$ are sought in the form

$$u = e^{ct}v(x, t), \quad v(x, t) = v(x, t - \tau), \quad (17)$$

where c is an arbitrary constant and $v(x, t)$ is a τ -periodic function. In this case, $w/u = p_0 = e^{-c\tau}$.

Substituting (17) into equation (10) yields a linear problem for determining v :

$$v_t = kv_{xx} + bv, \quad v(x, t) = v(x, t - \tau), \quad (18)$$

where $b = f(e^{-c\tau}) - c$.

The general solution to problem (18), which will be denoted $v = V_1(x, t; b)$ for convenience, is expressed as

$$V_1(x, t; b) = \sum_{n=0}^{\infty} \exp(-\lambda_n x) [A_n \cos(\beta_n t - \gamma_n x) + B_n \sin(\beta_n t - \gamma_n x)] \\ + \sum_{n=1}^{\infty} \exp(\lambda_n x) [C_n \cos(\beta_n t + \gamma_n x) + D_n \sin(\beta_n t + \gamma_n x)], \quad (19)$$

$$\beta_n = \frac{2\pi n}{\tau}, \quad \lambda_n = \left(\frac{\sqrt{b^2 + \beta_n^2} - b}{2k} \right)^{1/2}, \quad \gamma_n = \left(\frac{\sqrt{b^2 + \beta_n^2} + b}{2k} \right)^{1/2}, \quad (20)$$

where A_n , B_n , C_n , and D_n are arbitrary constants such that the series (19) with (20) as well as the derivatives $(V_1)_t$ and $(V_1)_{xx}$ are convergent; for example, the convergence can be ensured if one sets $A_n = B_n = C_n = D_n = 0$ for $n > N$, where N is some arbitrary positive integer.

The following special cases can be distinguished:

(i) τ -periodic (in t) solutions to problem (18) decaying as $x \rightarrow \infty$ are given by formulas (19) and (20) with $A_0 = B_0 = 0$, $C_n = D_n = 0$, and $n = 1, 2, \dots$;

(ii) τ -periodic solutions to problem (18) bounded as $x \rightarrow \infty$ are given by formulas (19) and (20) with $C_n = D_n = 0$ and $n = 1, 2, \dots$;

(iii) a stationary solution is given by formulas (19) and (20) with $A_n = B_n = C_n = D_n = 0$ and $n = 1, 2, \dots$.

To sum up, we have an exact solution to equation (10)

$$u = e^{ct}V_1(x, t; b), \quad b = f(e^{-c\tau}) - c, \quad (21)$$

where c is an arbitrary constant and $V_1(x, t; b)$ is a τ -periodic function determined by formulas (19) and (20).

1.3. Solutions of the difference equation (16) with $p_0 < 0$ are sought in the form

$$u = e^{ct}v(x, t), \quad v(x, t) = -v(x, t - \tau), \quad (22)$$

where c is an arbitrary constant and $v(x, t)$ is a τ -antiperiodic function. In this case, $w/u = p_0 = -e^{-c\tau}$.

Substituting (22) into (10) yields a linear problem for determining v :

$$v_t = kv_{xx} + bv, \quad v(x, t) = -v(x, t - \tau), \quad (23)$$

where $b = f(-e^{-c\tau}) - c$.

The general solution to problem (23), which will be denoted $v = V_2(x, t; b)$ for convenience, is expressed as

$$\begin{aligned} V_2(x, t; b) = & \sum_{n=1}^{\infty} \exp(-\lambda_n x) [A_n \cos(\beta_n t - \gamma_n x) + B_n \sin(\beta_n t - \gamma_n x)] \\ & + \sum_{n=1}^{\infty} \exp(\lambda_n x) [C_n \cos(\beta_n t + \gamma_n x) + D_n \sin(\beta_n t + \gamma_n x)], \end{aligned} \quad (24)$$

$$\beta_n = \frac{\pi(2n-1)}{\tau}, \quad \lambda_n = \left(\frac{\sqrt{b^2 + \beta_n^2} - b}{2k} \right)^{1/2}, \quad \gamma_n = \left(\frac{\sqrt{b^2 + \beta_n^2} + b}{2k} \right)^{1/2}, \quad (25)$$

where A_n, B_n, C_n , and D_n are arbitrary constants such that the series (24) with (25) as well as the derivatives $(V_2)_t$ and $(V_2)_{xx}$ are convergent. Solutions to problem (23) decaying as $x \rightarrow \infty$ and τ -antiperiodic in t are given by formulas (24) and (25) with $C_n = D_n = 0$ and $n = 1, 2, \dots$.

To sum up, we have an exact solution to equation (10):

$$u = e^{ct}V_2(x, t; b), \quad b = f(-e^{-c\tau}) - c, \quad (26)$$

where c is an arbitrary constant and $V_2(x, t; b)$ is a τ -antiperiodic function determined by formulas (24) and (25).

Remark 6. Solutions (19)–(20) and (24)–(25) look very similar. However, the summation in the former solution starts with $n = 0$ rather than $n = 1$ and the expressions of β_n are different.

4 Equations with one arbitrary function dependent on a linear combination of u and w

Equation 2. Consider the equation

$$u_t = ku_{xx} + bu + f(u - w), \quad (27)$$

which is a special case of equation (5) with $f(z) = b$, $g = 0$, and $z = u - w$ (the function h has been renamed f).

2.1. In this case, the functional constraint of the second kind (8) has the form

$$u - w = q(t), \quad w = u(x, t - \tau). \quad (28)$$

It is clear that the additive separable solution

$$u = \varphi(x) + \psi(t) \quad (29)$$

satisfies the difference equation (28). We have $q(t) = \psi(t) - \psi(t - \tau)$. Substituting (29) into (27) and separating the variables, one arrives at equations for determining $\varphi(x)$ and $\psi(t)$:

$$k\varphi''_{xx} + b\varphi = a, \quad (30)$$

$$\psi'_t(t) = b\psi(t) + a + f(\psi(t) - \psi(t - \tau)), \quad (31)$$

where a is an arbitrary constant.

The general solution to equation (30) with $b \neq 0$ and $a = 0$ is expressed as

$$\varphi(x) = \begin{cases} C_1 \cos(\alpha x) + C_2 \sin(\alpha x), & \alpha = \sqrt{b/k} \quad \text{if } b > 0; \\ C_1 \exp(-\alpha x) + C_2 \exp(\alpha x), & \alpha = \sqrt{-b/k} \quad \text{if } b < 0, \end{cases} \quad (32)$$

where C_1 and C_2 are arbitrary constants. Solution (29) with $b > 0$ is periodic in the space coordinate x .

The general solution to equation (30) with $b = 0$ and $a \neq 0$ is

$$\varphi(x) = \frac{a}{2k} x^2 + C_1 x + C_2. \quad (33)$$

Remark 7. Solutions of the form (29) are also admitted by the more general equation

$$u_t = ku_{xx} + bu + f(t, u - w),$$

in which the kinetic function is additionally dependent on t .

2.2. The functional constraint of the first kind (7) for equation (27) has the form

$$u - w = p(x), \quad w = u(x, t - \tau). \quad (34)$$

The difference equation (28) can be satisfied, for example, by choosing the generalized separable solution

$$u = t\varphi(x) + \psi(x), \quad (35)$$

which results in $p(x) = \tau\varphi(x)$.

Substituting (35) into (27) yields ordinary differential equations for $\varphi(x)$ and $\psi(x)$:

$$k\varphi''_{xx} + b\varphi = 0, \quad (36)$$

$$k\psi''_{xx} + b\psi + f(\tau\varphi) - \varphi = 0. \quad (37)$$

Equation (36) coincides with (30) at $a = 0$; its solution is given by formulas (32). The nonhomogeneous constant-coefficient linear ordinary differential equation (37) is easy to integrate.

Using the above solutions (29) and (35) as well as the theorem below, one can obtain more complex exact solutions to equation (27); these solutions can have any number of arbitrary parameters.

Theorem 1 (nonlinear superposition of solutions). *Suppose $u_0(x, t)$ is a solution to the nonlinear equation (27) and $v = V_1(x, t; b)$ is any τ -periodic solution to the linear heat equation with source (18). Then the sum*

$$u = u_0(x, t) + V_1(x, t; b) \quad (38)$$

is also a solution to equation (27). The general form of the function $V_1(x, t; b)$ is given by formulas (19) and (20).

Remark 8. For the nonlinear equation (27), the particular solution $u_0(x, t)$ in formula (38) can be taken in the traveling-wave form $u_0 = u_0(\alpha x + \beta t)$.

Remark 9. Formula (38) also remains valid for the more general equation

$$u_t = ku_{xx} + bu + f(x, t, u - w),$$

in which the kinetic function depends on three arguments. If $f = f(x, u - w)$, then a stationary solution $u_0 = u_0(x)$ can be used as the first term in (38), while the second term can be taken in the form (19)–(20). If $f = f(t, u - w)$, then a spatially homogeneous solution $u_0 = u_0(t)$ can be taken for the first term in (38).

Equation 3. Consider the equation

$$u_t = ku_{xx} + bu + f(u - aw), \quad a > 0, \quad (39)$$

which is a special case of equation (5) with $f(z) = b$, $g = 0$, and $z = u - aw$; for convenience, the function h has been renamed f .

3.1. The functional constraint of the first kind (7) for equation (27) has the form

$$u - aw = p(x), \quad w = u(x, t - \tau). \quad (40)$$

The difference equation (40) can be satisfied, for example, with the generalized separable solution

$$u = e^{ct}\varphi(x) + \psi(x), \quad c = \frac{1}{\tau} \ln a, \quad (41)$$

which gives $p(x) = (1 - a)\psi(x)$.

Substituting (41) into (39) leads to ordinary differential equations for $\varphi(x)$ and $\psi(x)$:

$$k\varphi'' + (b - c)\varphi = 0, \quad (42)$$

$$k\psi'' + b\psi + f(\eta) = 0, \quad \eta = (1 - a)\psi. \quad (43)$$

Up to obvious renaming of variables, equation (42) coincides with (30) at $a = 0$; its solution is given by formulas (32) where b must be substituted by $b - c$.

3.2. Using the above solution (41)–(43) as well as the theorem below, one can obtain more complex exact solutions to equation (39); these solutions can have any number of arbitrary free parameters.

Theorem 2 (generalization of Theorem 1). *Suppose $u_0(x, t)$ is a solution to the nonlinear equation (39) and $v = V_1(x, t; b)$ is any τ -periodic solution to the linear heat equation with source (18). Then the sum*

$$u = u_0(x, t) + e^{ct}V_1(x, t; b - c), \quad c = \frac{1}{\tau} \ln a, \quad (44)$$

is also a solution to equation (39). The general form of the function $V_1(x, t; b)$ is given by formulas (19) and (20).

Formula (44) makes it possible to obtain a wide class of exact solutions to the nonlinear equation (39). Apart from (41), one can take $u_0 = u_0(x)$ and $u_0 = u_0(t)$ as well as the more general, traveling wave solution $u_0 = \theta(\alpha x + \beta t)$ as the particular solution $u_0 = u_0(x, t)$; the constants α and β are arbitrary and the function $\theta(y)$ satisfies the ordinary differential-difference equation

$$k\alpha^2\theta''(y) - \beta\theta'(y) + b\theta(y) + f(\theta(y) - a\theta(y - \sigma)) = 0, \quad y = \alpha x + \beta t, \quad \sigma = \beta\tau.$$

Remark 10. Formula (44) also remains valid for the more general equation

$$u_t = ku_{xx} + bu + f(x, t, u - aw),$$

in which the kinetic function depends on three arguments.

Equation 4. Consider the equation

$$u_t = ku_{xx} + bu + f(u + aw), \quad a > 0, \quad (45)$$

which is a special case of equation (5) with $f(z) = b$, $g = 0$, and $z = u + aw$; for convenience, the function h has been renamed f .

Theorem 3. Suppose $u_0(x, t)$ is a solution to the nonlinear equation (45) and $v = V_2(x, t; b)$ is any τ -antiperiodic solution to the linear heat equation with source (23). Then the sum

$$u = u_0(x, t) + e^{ct}V_2(x, t; b - c), \quad c = \frac{1}{\tau} \ln a, \quad (46)$$

is also a solution to equation (45). The general form of the function $V_2(x, t; b)$ is given by formulas (24) and (25).

Formula (46) makes it possible to obtain a wide class of exact solutions to the nonlinear equation (45). One can take $u_0 = u_0(x)$ and $u_0 = u_0(t)$ as well as the more general, traveling wave solution $u_0 = \theta(\alpha x + \beta t)$ as the particular solution $u_0(x, t)$; the constants α and β are arbitrary and the function $\theta(y)$ satisfies the ordinary differential-difference equation

$$k\alpha^2\theta''(y) - \beta\theta'(y) + b\theta(y) + f(\theta(y) + a\theta(y - \sigma)) = 0, \quad y = \alpha x + \beta t, \quad \sigma = \beta\tau.$$

Remark 11. Formula (46) also remains valid for the more general equation

$$u_t = ku_{xx} + bu + f(x, t, u + aw),$$

in which the kinetic function depends on three arguments.

5 Equations with two arbitrary functions dependent on a linear combination of u and w

Equation 5. Now consider the more general equation

$$u_t = ku_{xx} + uf(u - w) + wg(u - w) + h(u - w), \quad (47)$$

where $f(z)$, $g(z)$, and $h(z)$ are arbitrary functions; in this case, either function f or g can be set equal to zero without loss of generality.

5.1. The difference constraint (7) for equation (47) has the form (34). The linear difference equation (34) can be satisfied, as previously, with a generalized separable solution of the form (35). As a result, one can obtain equations for determining $\varphi(x)$ and $\psi(x)$; these equations are not written out, since a significantly more general result will be presented below.

5.2. The linear difference equation (34) can be satisfied by setting

$$u = \sum_{n=1}^N [\varphi_n(x) \cos(\beta_n t) + \psi_n(x) \sin(\beta_n t)] + t\theta(x) + \xi(x), \quad \beta_n = \frac{2\pi n}{\tau}, \quad (48)$$

where N is an arbitrary positive integer. In this case, we have $p(x) = \tau\varphi(x)$ on the right-hand side of equation (34).

Substituting (48) into (47) and performing simple rearrangements, we obtain

$$\sum_{n=1}^N [A_n \cos(\beta_n t) + B_n \sin(\beta_n t)] + Ct + D = 0, \quad (49)$$

where the functional coefficients A_n , B_n , C , and D are dependent on $\varphi_n(x)$, $\psi_n(x)$, $\theta(x)$, and $\xi(x)$ as well as their derivatives and independent of t . In (49), equating all the functional coefficients with zero,

$A_n = B_n = C = D = 0$, we arrive at the following ordinary differential equation for the unknown functions:

$$\begin{aligned} k\varphi_n'' + \varphi_n[f(\tau\theta) + g(\tau\theta)] - \beta_n\psi_n &= 0, \\ k\psi_n'' + \psi_n[f(\tau\theta) + g(\tau\theta)] + \beta_n\varphi_n &= 0, \\ k\theta'' + \theta[f(\tau\theta) + g(\tau\theta)] &= 0, \\ k\xi'' + \xi f(\tau\theta) + (\xi - \tau\theta)g(\tau\theta) + h(\tau\theta) - \theta &= 0. \end{aligned}$$

Note that the third nonlinear equation admits the trivial particular solution $\theta = 0$; in this case, all other equations become linear with constant coefficients.

Remark 12. Solutions of the form (48) are also admitted by the more general equation

$$u_t = ku_{xx} + uf(x, u - w) + wg(x, u - w) + h(x, u - w).$$

Equation 6. Consider the equation

$$u_t = ku_{xx} + uf(u - aw) + wg(u - aw) + h(u - aw), \quad (50)$$

where $f(z)$, $g(z)$, and $h(z)$ are arbitrary functions, which generalizes equation (39).

6.1. The difference constraint (7) for equation (50) has the form (40). The linear difference equation (40) can be satisfied, as previously, with a generalized separable solution of the form (41). As a result, one arrives at equations for determining $\varphi(x)$ and $\psi(x)$; these equations are not written here, since a much more general result is given below.

6.2. The linear difference equation (34) can be satisfied by setting

$$\begin{aligned} u &= e^{ct} \left\{ \theta(x) + \sum_{n=1}^N [\varphi_n(x) \cos(\beta_n t) + \psi_n(x) \sin(\beta_n t)] \right\} + \xi(x), \\ c &= \frac{1}{\tau} \ln a, \quad \beta_n = \frac{2\pi n}{\tau}, \end{aligned} \quad (51)$$

where N is an arbitrary positive integer. In this case, we have $p(x) = (1 - a)\xi(x)$ of the right-hand side of equation (40).

Substituting (51) into (50) and reasoning in a similar fashion as for equation (47), we arrive at the following equations for determining $\theta(x)$, $\varphi_n(x)$, $\psi_n(x)$, and $\xi(x)$:

$$\begin{aligned} k\theta'' + \theta \left[f(\eta) + \frac{1}{a}g(\eta) - c \right] &= 0, \quad \eta = (1 - a)\xi, \\ k\varphi_n'' + \varphi_n \left[f(\eta) + \frac{1}{a}g(\eta) - c \right] - \beta_n\psi_n &= 0, \\ k\psi_n'' + \psi_n \left[f(\eta) + \frac{1}{a}g(\eta) - c \right] + \beta_n\varphi_n &= 0, \\ k\xi'' + \xi[f(\eta) + g(\eta)] + h(\eta) &= 0. \end{aligned}$$

Remark 13. Solutions of the form (51) are also admitted by the more general equation

$$u_t = ku_{xx} + uf(x, u - aw) + wg(x, u - aw) + h(x, u - aw), \quad a > 0.$$

Equation 7. Consider the equation

$$u_t = ku_{xx} + uf(u + aw) + wg(u + aw) + h(u + aw), \quad a > 0, \quad (52)$$

where $f(z)$, $g(z)$, and $h(z)$ are arbitrary functions, which generalizes equation (45).

The difference constraint (7) for equation (52) has the form

$$u + aw = p(x), \quad w = u(x, t - \tau). \quad (53)$$

The linear difference equation (53) can be satisfied by setting

$$u = e^{ct} \sum_{n=1}^N [\varphi_n(x) \cos(\beta_n t) + \psi_n(x) \sin(\beta_n t)] + \xi(x), \quad (54)$$

$$c = \frac{1}{\tau} \ln a, \quad \beta_n = \frac{\pi(2n+1)}{\tau},$$

where N is an arbitrary positive integer. In this case, we have $p(x) = (1+a)\xi(x)$ on the right-hand side of equation (53).

Substituting (54) into (52) and reasoning in a similar fashion as for equation (47), we arrive at the following equations for determining $\varphi_n(x)$, $\psi_n(x)$, and $\xi(x)$:

$$\begin{aligned} k\varphi_n'' + \varphi_n \left[f(\eta) - \frac{1}{a}g(\eta) - c \right] - \beta_n \psi_n &= 0, \\ k\psi_n'' + \psi_n \left[f(\eta) - \frac{1}{a}g(\eta) - c \right] + \beta_n \varphi_n &= 0, \\ k\xi'' + \xi[f(\eta) + g(\eta)] + h(\eta) &= 0, \quad \eta = (1+a)\xi. \end{aligned}$$

The last equation is independent.

Remark 14. Solutions of the form (54) are also admitted by the more general equation

$$u_t = ku_{xx} + uf(x, u + aw) + wg(x, u + aw) + h(x, u + aw), \quad a > 0.$$

6 Equation with two arbitrary functions dependent on $u^2 + w^2$

Equation 8. Now consider the equation

$$u_t = ku_{xx} + uf(u^2 + w^2) + wg(u^2 + w^2), \quad (55)$$

which contains a nonlinear (quadratic) argument, $z = u^2 + w^2$.

The difference constraint (7) for equation (55) has the form

$$u^2 + w^2 = p(x), \quad w = u(x, t - \tau). \quad (56)$$

The nonlinear difference equation (53) can be satisfied by setting

$$\begin{aligned} u &= \varphi_n(x) \cos(\lambda_n t) + \psi_n(x) \sin(\lambda_n t), \\ \lambda_n &= \frac{\pi(2n+1)}{2\tau}, \quad n = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (57)$$

It is not difficult to verify that

$$w = (-1)^n \varphi_n(x) \sin(\lambda_n t) + (-1)^{n+1} \psi_n(x) \cos(\lambda_n t)$$

and

$$u^2 + w^2 = \varphi_n^2(x) + \psi_n^2(x) = p(x).$$

Substituting (57) into (55) and splitting the resulting expression with respect to $\cos(\lambda_n t)$ and $\sin(\lambda_n t)$, we arrive at a nonlinear system of ordinary differential equations for determining $\varphi_n(x)$ and $\psi_n(x)$:

$$\begin{aligned} k\varphi_n'' + \varphi_n f(\varphi_n^2 + \psi_n^2) + (-1)^{n+1} \psi_n g(\varphi_n^2 + \psi_n^2) - \lambda_n \psi_n &= 0, \\ k\psi_n'' + \psi_n f(\varphi_n^2 + \psi_n^2) + (-1)^n \varphi_n g(\varphi_n^2 + \psi_n^2) + \lambda_n \varphi_n &= 0. \end{aligned}$$

Remark 15. Solutions of the form (54) are also admitted by the more general equation

$$u_t = ku_{xx} + uf(x, u^2 + w^2) + wg(x, u^2 + w^2).$$

7 More general nonlinear delay partial differential equations

Now consider nonlinear partial differential-difference equations of the more general form

$$\begin{aligned} L[u] &= M[u] + uf(z) + wg(z) + h(z), \\ w &= u(x, t - \tau), \quad z = z(u, w), \end{aligned} \quad (58)$$

where L and M are arbitrary constant-coefficient linear differential operators with respect to t and x :

$$L[u] = \sum_{i=1}^k b_i \frac{\partial^i u}{\partial t^i}, \quad M[u] = \sum_{i=1}^m a_i \frac{\partial^i u}{\partial x^i}. \quad (59)$$

As before, $f(z)$, $g(z)$, and $h(z)$ are arbitrary functions and $z = z(u, w)$ is a given function.

By setting $L[u] = u_{tt}$ and $M[u] = a^2 u_{xx}$ in (58), we get the nonlinear delay Klein–Gordon equation (a delay hyperbolic PDE)

$$u_{tt} = a^2 u_{xx} + uf(z) + wg(z) + h(z). \quad (60)$$

Remark 16. Oscillation properties of some hyperbolic partial differential equations with delay were studied, for example, in [40–42].

Many of the results obtained previously for nonlinear delay reaction–diffusion equations (5) also apply to nonlinear delay partial differential equations of the general form (58)–(59).

Subsequently, in deriving determining equations, we use the following simple properties of the operators L and M :

$$\begin{aligned} L[\varphi(x)\psi(t)] &= \varphi(x)L[\psi(t)], & L[\varphi(x) + \psi(t)] &= L[\psi(t)], \\ M[\varphi(x)\psi(t)] &= \psi(t)M[\varphi(x)], & M[\varphi(x) + \psi(t)] &= M[\varphi(x)]. \end{aligned}$$

Below we omit intermediate steps and give only final results.

Equation 9. The equation

$$L[u] = M[u] + uf(w/u), \quad (61)$$

admits the multiplicative separable solution

$$u = \varphi(x)\psi(t), \quad (62)$$

with the functions $\varphi(x)$ and $\psi(t)$ satisfying the linear ordinary differential and differential-difference equations

$$M[\varphi] = c\varphi, \quad (63)$$

$$L[\psi(t)] = c\psi(t) + \psi(t)f(t, \psi(t - \tau)/\psi(t)), \quad (64)$$

where c is an arbitrary constant.

It should be noted that:

- (i) if $c = 0$, equation (63) admits polynomial particular solutions;
- (ii) for any c , equation (63) admits exponential particular solutions $\varphi(x) = Ae^{\beta x}$, where A is an arbitrary constant and β is a root of the polynomial equation $\sum_{i=1}^m a_i \beta^i = c$;
- (iii) equation (64) admits exponential particular solutions $\psi(t) = Be^{\lambda t}$, where B is an arbitrary constant and λ is a root of the transcendental equation

$$\sum_{i=1}^n b_i \lambda^i = c + f(e^{-\tau\lambda}).$$

Equation 10. The equation

$$L[u] = M[u] + bu + f(u - w) \quad (65)$$

admits additive separable solutions of the form

$$u = \varphi(x) + \psi(t), \quad (66)$$

with the functions $\varphi(x)$ and $\psi(t)$ described by the linear partial differential equation and ordinary differential-difference equation

$$M[\varphi] = c - b\varphi, \quad (67)$$

$$L[\psi(t)] = c + b\psi(t) + f(\psi(t) - \psi(t - \tau)) \quad (68)$$

and c being an arbitrary constant.

It should be noted that:

(i) if $b = 0$, equation (67) admits polynomial particular solutions;

(ii) if $c = 0$, equation (67) admits exponential particular solutions $\varphi(x) = Ae^{\beta x}$, where A is an arbitrary constant and β is a root of the polynomial equation $\sum_{i=1}^n a_i \beta^i = -b$;

Using the above solution (66) and the theorem below, one can find more complex exact solutions to equation (65); these solutions can have any number of arbitrary free parameters.

Theorem 4 (nonlinear superposition of solutions). *Suppose $u_0(x, t)$ is a solution to the nonlinear equation (65) and $v = v(x, t)$ is any τ -periodic solution to the linear equation*

$$L[v] = M[v] + bv, \quad v(x, t) = v(x, t - \tau). \quad (69)$$

Then the sum

$$u = u_0(x, t) + v(x, t) \quad (70)$$

is also solution to equation (65).

Apart from (66), the particular solution $u_0(x, t)$ can be sought in the traveling-wave form $u_0 = \theta(\alpha x + \beta t)$. Particular solutions to the linear problem (69) can be sought in the form

$$\begin{aligned} v(x, t) = & \sum_{n=0}^N \exp(-\lambda_n x) [A_n \cos(\beta_n t - \gamma_n x) + B_n \sin(\beta_n t - \gamma_n x)] \\ & + \sum_{n=1}^N \exp(\lambda_n x) [C_n \cos(\beta_n t + \gamma_n x) + D_n \sin(\beta_n t + \gamma_n x)], \end{aligned} \quad (71)$$

$$\beta_n = 2\pi n / \tau,$$

where N is any positive integer, A_n, B_n, C_n , and D_n are arbitrary constants, and λ_n and γ_n are constants determined from substituting (71) into equation (69).

Equation 11. For the equation

$$L[u] = M[u] + bu + f(u - aw), \quad a > 0, \quad (72)$$

the following theorem holds.

Theorem 5 (generalizes Theorem 4). *Suppose $u_0(x, t)$ is a solution to the nonlinear equation (72) and $v = v(x, t)$ is any τ -periodic solution to the linear equation*

$$L_1[v] = M[v] + bv, \quad v(x, t) = v(x, t - \tau), \quad (73)$$

where $L_1[v] \equiv e^{-ct} L[e^{ct} v]$ is a linear differential operator with constant coefficients. Then the sum

$$u = u_0(x, t) + e^{ct} v(x, t), \quad c = \frac{1}{\tau} \ln a, \quad (74)$$

is also a solution to equation (72).

Formula (74) makes it possible to obtain a wide class of exact solutions to the nonlinear equation (72). Simple solutions of the form $u_0 = u_0(x)$ and $u_0 = u_0(t)$ as well as the more general, traveling wave solution $u_0 = \theta(\alpha x + \beta t)$ can be taken as the particular solution $u_0(x, t)$. Just as before, particular solutions to the linear problem (73) can be sought in the form (71).

Equation 12. For the equation

$$L[u] = M[u] + bu + f(u + aw), \quad a > 0, \quad (75)$$

the following theorem holds.

Theorem 6. Suppose $u_0(x, t)$ is a solution to the nonlinear equation (75) and $v = v(x, t)$ is any τ -antiperiodic solution to the linear equation

$$L_1[v] = M[v] + bv, \quad v(x, t) = -v(x, t - \tau), \quad (76)$$

where $L_1[v] \equiv e^{-ct}L[e^{ct}v]$ is a linear differential operator with constant coefficients. Then the sum

$$u = u_0(x, t) + e^{ct}v(x, t), \quad c = \frac{1}{\tau} \ln a, \quad (77)$$

is also a solution to equation (75).

Formula (77) makes it possible to obtain a wide class of exact solutions to the nonlinear equation (75). Simple solutions of the form $u_0 = u_0(x)$ and $u_0 = u_0(t)$ as well as the more general, traveling wave solution $u_0 = \theta(\alpha x + \beta t)$ can be taken as the particular solution $u_0(x, t)$. Particular solutions to the linear problem (76) can be sought in the form

$$\begin{aligned} v(x, t) &= \sum_{n=1}^N \exp(-\lambda_n x) [A_n \cos(\beta_n t - \gamma_n x) + B_n \sin(\beta_n t - \gamma_n x)] \\ &\quad + \sum_{n=1}^N \exp(\lambda_n x) [C_n \cos(\beta_n t + \gamma_n x) + D_n \sin(\beta_n t + \gamma_n x)], \\ \beta_n &= \frac{\pi(2n-1)}{\tau}, \end{aligned} \quad (78)$$

where N is any positive integer, A_n, B_n, C_n , and D_n are arbitrary constants, and λ_n and γ_n are constants determined from substituting (78) into (76).

Equation 13. Exact solutions to the nonlinear delay partial differential equation

$$L[u] = M[u] + uf(x, u - w) + wg(x, u - w) + h(x, u - w) \quad (79)$$

are sought in the form (48).

Equation 14. Exact solutions to the nonlinear delay partial differential equation

$$L[u] = M[u] + uf(x, u - aw) + wg(x, u - aw) + h(x, u - aw), \quad a > 0, \quad (80)$$

are sought in the form (51).

Equation 15. Exact solutions to the nonlinear delay partial differential equation

$$L[u] = M[u] + uf(x, u + aw) + wg(x, u + aw) + h(x, u + aw), \quad a > 0, \quad (81)$$

are sought in the form (54).

Equation 16. Exact solutions to the nonlinear delay partial differential equation

$$L[u] = M[u] + uf(x, u^2 + w^2) + wg(x, u^2 + w^2) \quad (82)$$

are sought in the form (57).

Remark 17. In equations (79), (80), (81), and (82), the linear operator M can depend on x and have the form

$$M[u] = \sum_{i=1}^m a_i(x) \frac{\partial^i u}{\partial x^i},$$

where $a_i(x)$ are arbitrary functions.

Remark 18. In equations (79), (80), (81), and (82), the functions f , g , and h can explicitly depend on several space coordinates, $x = (x_1, \dots, x_s)$, and M can be an arbitrary linear differential operator of any order (first, second, or higher) with respect to the space coordinates whose coefficients can be explicitly dependent on $x = (x_1, \dots, x_s)$. Just as before, exact solutions to such equations are sought in the form (48), (51), (54), and (57), respectively, where x is understood as $x = (x_1, \dots, x_s)$.

In particular, one can choose an elliptic operator,

$$M[u] = \sum_{i,j=1}^s \frac{\partial}{\partial x_i} k_{ij}(\mathbf{x}) \frac{\partial u}{\partial x_j} + \sum_{i=1}^s c_i(\mathbf{x}) \frac{\partial u}{\partial x_i},$$

which arises in the studies [20, 21] in the case $c_i = 0$.

8 General partial differential equations with time-varying delay

In very much the same manner, one can obtain exact solutions to some nonlinear partial functional differential equations with time-varying delay of general form.

Below we give a few simple examples to illustrate the aforesaid.

Equation 17. Consider nonlinear partial functional differential equations of the form

$$L[u] = M[u] + uf(w/u), \quad w = u(x, \zeta(t)), \quad (83)$$

where $\zeta(t)$ is a given function. In applications (e.g., see [20]), it is conventional to write $\zeta(t) = t - \tau(t)$, where $\tau(t)$ is a time-varying delay such that $0 \leq \tau(t) < \tau_0$.

In this case, the functional constraint (8) becomes

$$w/u = q(t), \quad w = u(x, \zeta(t)). \quad (84)$$

The functional equation (84) can be satisfied with the simple multiplicative separable solution

$$u = \varphi(x)\psi(t), \quad (85)$$

which gives $q(t) = \psi(\zeta(t))/\psi(t)$. Substituting (85) into (83) and separating the variables, one arrives at the following equations for $\varphi(x)$ and $\psi(t)$:

$$\begin{aligned} M[\varphi] &= c\varphi, \\ L[\psi(t)] &= c\psi(t) + \psi(t)f(t, \psi(\zeta)/\psi(t)), \quad \zeta = \zeta(t), \end{aligned}$$

where c is an arbitrary constant.

Equation 18. Consider nonlinear partial functional differential equations of the form

$$L[u] = M[u] + bu + f(u - w), \quad w = u(x, \zeta(t)). \quad (86)$$

In this case, the functional constraint (8) becomes

$$u - w = q(t), \quad w = u(x, \zeta(t)). \quad (87)$$

The equation admits an additive separable solution of the form

$$u = \varphi(x) + \psi(t), \quad (88)$$

which gives $q(t) = \psi(t) - \psi(\zeta(t))$. Substituting (88) into (86) and separating the variables, one arrives at the following equations for $\varphi(x)$ and $\psi(t)$:

$$\begin{aligned} M[\varphi] &= c - b\varphi, \\ L[\psi(t)] &= c + b\psi(t) + f(\psi(t) - \psi(\zeta)), \quad \zeta = \zeta(t), \end{aligned}$$

where c is an arbitrary constant.

9 Brief conclusions

To summarize, we have proposed a new method, which we call the functional constraints method, for constructing exact solutions of nonlinear delay reaction–diffusion equations of the form

$$u_t = ku_{xx} + F(u, w),$$

where $u = u(x, t)$, $w = u(x, t - \tau)$, and τ is the relaxation time. The method is based on searching for generalized separable solutions of the form

$$u = \sum_{n=1}^N \xi_n(x) \eta_n(t),$$

with the functions $\xi_n(x)$ and $\eta_n(t)$ determined from additional functional constraints (difference or functional equations) and the original delay partial differential equation.

All of the equations considered contain one or two arbitrary functions of a single argument. We have described a considerable number of new exact generalized separable solutions and a few more complex solutions representing a nonlinear superposition of generalized separable solutions and traveling wave solutions.

All solutions involve free parameters (in some cases, infinitely many parameters) and so can be suitable for solving certain problems and testing approximate analytical and numerical methods for nonlinear delay PDEs.

The results have been extended to the following classes of equations:

(i) nonlinear delay partial differential equations of the form

$$L[u] = M[u] + F(u, w),$$

where L and M are arbitrary constant-coefficient linear differential operators of any order with respect to the independent variables x and t ; in particular, this broad class includes the nonlinear delay Klein–Gordon equation, which is a hyperbolic-type delay equation;

(ii) some nonlinear delay partial differential equations that are explicitly dependent on either of the independent variables, x or t ; and

(iii) some nonlinear partial functional differential equations with time-varying delay of general form.

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